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STEADY SOLITON-LIKE SOLUTIONS OF EULER'S EQUATIONS WITH INTRINSIC FORCE FIELDS†

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Steady solutions of a system of Euler's equations when there are intrinsic force interactions of electrical and gravitational type are analysed. Particular attention is devoted to the class of soliton "particle-like" solutions with concentrated charge and mass. Periodic steady solutions for the case when the intrinsic gravitational field has non-zero density at infinity is also considered.

Soliton solutions of the hydrodynamic equations have an extensive bibliography. It is sufficient here to mention the monographs and surveys on gravitational waves on the surface of a liquid [1–3] and ion-acoustic solitons in hydrodynamic models of a plasma [1, 2, 4]. The classical soliton solutions [5, 6] were obtained as exact continuous solutions of the non-linear wave or evolution equations. Particle-like solutions, which differ from the classical solutions, were obtained in [7, 8] as the steady solutions of equations of the two-fluid model of a cold plasma.

1. THE INITIAL EQUATIONS

We will write the initial equations of motion of an ideal fluid (Euler's equations) taking into account intrinsic internal interactions of the particles of the medium, which will be taken to be electrical interaction of charged particles or gravitational interaction of neutral particles. Assuming that the motions of the fluid occur adiabatically or isothermally, we shall not use the equation of conservation of energy.

The equations of continuity and conservation of momentum have the form

$$\frac{\partial n}{\partial t}+\operatorname{div}(n\mathbf{V})=0,\quad \frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\nabla)\mathbf{V}=-\frac{1}{nm}\nabla P+\mathbf{F} \tag{1.1}$$

Here m is the mass of the particles of the medium, n, P and $\mathbf{V}$ are the concentration, pressure and velocity of the medium respectively, and $\mathbf{F}$ is a function characterizing the force field associated with the particles of the medium.

We will formulate the problem for the two cases mentioned. A typical example of the first case is the equations of motion of an ionic gas in the hydrodynamic "two-fluid" model of a plasma, where the interaction between the ion and electron components of the plasma is weak [1, 4, 9]. For electrical interaction

$$\mathbf{F}=-\nabla\varphi_1 \tag{1.2}$$

where φ_1 is the electrical field potential, determined from Poisson's equation

$$\Delta\varphi_1=4\pi\frac{e^2}{m}(n_e-n) \tag{1.3}$$

Here e is the electron charge and n_e is the electron density.

System (1.1)–(1.3) is closed by the equation of state of an ion gas, for example, in the form

$$P=nT \tag{1.4}$$

where T is the constant temperature of the ion component of the plasma, and by specifying the electron density distribution, for example, the Boltzmann's distribution

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$$n_e = n_0 \exp(e\varphi_1 / T_e) \tag{1.5}$$

where T_e is the constant temperature of the electron component of the plasma and n_0 is the unperturbed particle density at infinity.

At low perturbation amplitudes, system (1.1)–(1.5) yields the Boussinesq equations [1]. Further linearization of these equations relative to "simple" wave solutions leads to the Korteweg–de Vries equation, the solutions of which are isolated waves—solitons. For finite perturbation amplitudes, we can use the method described in [10] to make a fuller study of the structure of steady solutions of system (1.1)–(1.5) in the one-dimensional case. In particular, "subsonic" soliton solutions with charge and mass concentrated at the centre of the soliton were obtained in [7, 8].

The soliton solutions of system (1.1)–(1.5) are studied in detail below. The analysis will be clearer if instead of using (1.5) we assume that $n_e = n_0$. This is valid, first, in regions where the magnitude of the potential φ_1 is close to zero and, second, in regions where φ_1 is negative and $n_e \ll n$. The right-hand side of Poisson's equation (1.3) is then a weak function of n_e and we can put $n_e = n_0$ without affecting the structure of the resulting steady solutions.

For the motion of neutral gravitating particles, we put

$$\mathbf{F} = \nabla\varphi_2 \tag{1.6}$$

where φ_2 is the gravitational field potential given by Poisson's equation

$$\Delta\varphi_2 = 4\pi G m(n_0 - n) \tag{1.7}$$

(G is the gravitational constant). The form of the right-hand side of Eq. (1.7) enables us to take into account the "repulsive" force [11] which operates at large cosmic distances and partially neutralizes the force of gravitational attraction.

The system of equations (1.1), (1.6) and (1.7) is also closed.

Below we will analyse the steady solutions of the one-dimensional system of equations which follow from (1.1) and (1.2)

$$\frac{\partial n}{\partial t} + \frac{\partial nu}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{c^2}{n}\frac{\partial n}{\partial x} + F \tag{1.8}$$

where u is the velocity component of the medium projected onto the x axis and $c^2 = T/m$ is the characteristic velocity of sound (the velocity of propagation of small perturbations).

System (1.8) is completed by the following relations:

for a medium of charged particles with electrostatic interaction (with the above assumptions)

$$F = -\partial\varphi_1 / \partial x, \quad \Delta\varphi_1 = b_1^2(1 - n) \tag{1.9}$$

for a medium of neutral gravitating particles

$$F = \partial\varphi_2 / \partial x, \quad \Delta\varphi_2 = b_2^2(1 - n) \tag{1.10}$$

All the quantities in (1.8)–(1.10) and below are assumed to be dimensionless, the ion density n being taken relative to its unperturbed value n_0, the velocities u and c relative to the characteristic value u_0, the coordinate x relative to x_0, the time t relative to t_0 and the potentials φ and φ_2 relative to $\varphi_0 = u^2$. Then the positive constants b_1 and b_2 are defined as follows:

$$b_1^2 = \frac{4\pi e^2 n_0}{m u_0^2}, \quad b_2^2 = \frac{4\pi G m n_0}{u_0^2} \tag{1.11}$$

We will now consider the structure of the steady solutions of the system of equations (1.8), (1.9) for a medium of charged particles with an intrinsic electrostatic field and system (1.8), (1.10) for a medium of neutral gravitating particles.

We will consider the steady solutions of the initial equations in the system of coordinates (X, t), where t is the same as before, $X = x + Ut$ and U is the constant velocity of motion of the new system of coordinates relative to the old. Changing to variables (X, t) and integrating the equation of continuity with respect to x from (1.8) once, we obtain

$$n(u + U) = A, \quad (u + U)\frac{du}{dX} = -c^2\frac{d\ln n}{dX} + F \tag{1.12}$$

The constant of integration A will be determined from the boundary conditions as $X \to \pm\infty$ in each specific case. From the second equation of system (1.12) we obtain the relation

$$\frac{(u+U)^2}{2} + c^2 \ln n + \int F dX = \text{const} \tag{1.13}$$

which could be called the analogue of the Bernoulli integral.

2. THE STRUCTURE OF STEADY SOLUTIONS FOR A MEDIUM OF CHARGED PARTICLES

Consider the steady solutions of system (1.9), (1.12) which satisfy the boundary conditions $n \to n_0 = 1$, $u \to 0$ as $X \to \pm\infty$. Then the constant A in the first equation of system (1.12) is equal to U. We differentiate the second equation of (1.12) with respect to X and eliminate the density $n = U/(u + U)$ and potential φ_1 from the resulting relation using (1.9). Taking $\upsilon = u + U$ as the independent variable and $p = d\upsilon / dX$ as the dependent variable, we obtain the first-order equation

$$\frac{dp}{d\upsilon} = -\frac{p^2(\upsilon^2 + c^2) + b_1^2(\upsilon - U)\upsilon}{(\upsilon^2 - c^2)p\upsilon} \tag{2.1}$$

This yields, in particular, a linear equation for the function $w(\upsilon) = p^2(\upsilon)$.

The pattern of integral curves of Eq. (2.1) in the phase plane (υ, p) is symmetric about the horizontal axis $p = 0$ and contains two singular points on that axis, with coordinates

$$\upsilon_1 = 0, \ \ p_1 = 0; \ \upsilon_2 = U, p_2 = 0 \tag{2.2}$$

The first singular point is a degenerate node. The integral curves touch the vertical axis at that point when $\upsilon \geqslant 0$. Near that point, they have the asymptotic formula

$$p^2 = \frac{2b_1^2}{c^2}(C\upsilon^2 + \upsilon^2 \ln|\upsilon| + U\upsilon)$$

where C is a constant (a parameter which defines the corresponding integral curve near the singular point).

The second singular point is a saddle if $U^2 \leqslant c^2$ and a centre if $U^2 > c^2$. In the former case the directions of approach of the separatrix towards that point are given by the angular coefficients

$$k_{1,2} = \pm b_1\sqrt{c^2 - U^2}$$

In the physical plane, motion along the separatrix from the second singular point corresponds to motion from infinity (with respect to X). Near the second singular point, we have the following asymptotic relation (when $U^2 < c^2$)

$$p^2 = b_1^2(\upsilon - U)^2 / (c^2 - U^2)$$

Two other singular points can exist in the phase plan with coordinates

$$\upsilon_{3,4} = c, \ \ p_{3,4} = \pm b_1\sqrt{(U-c)/(2c)} \tag{2.3}$$

When $U > c$, these are saddle singular points. When $U < c$, their ordinates are imaginary. Motion towards them occurs along the vertical separatrix and motion away from them occurs along the separatrices with angular coefficients

$$k_{3,4} = \mp[b_1/(6c^3)]\sqrt{2c/(U-c)}$$

In the physical plane, the latter corresponds to motion from infinity. Near the third and fourth singular points, we have the asymptotic formula

$$p = b_1\sqrt{(U-c)/(2c)} + [b_1/(6c^3)]\sqrt{2c/(U-c)}(c - \upsilon)$$

When $U = c$ the second, third and fourth singular points coincide. Motion towards this singular point is along the vertical separatrix $\upsilon = c$ and motion away from it is along the separatrix

$$p^2 = -[b_1 /(3c)](\upsilon - c)$$

The last singular point with negative abscissa $\upsilon = -c$ can only be considered when $U < 0$, which yields the same situation as before.

Figure 1 shows the structure of the solutions in the phase plane for $c = 1, b_1 = 9$ and various values of U: $U = 0.5$ (a), $U = 1$ (b) and $U = 1.5$ (c). The bold curves with arrows are separatrices passing through the singular points.

The solution in the form of a subsonic particle-like soliton (the separatrix in Fig. 1a joining the first and second singular points) is similar in character to the "subsonic" solitons considered in detail in [7, 8]. Note that a change of the coordinate X from $+\infty$ to $-\infty$ corresponds to moving along the separatrix from the second singular point in the direction shown by the arrows through the first nodal singular point where $X = 0$ to the second singular point (on the bottom of the phase plane). For the soliton solution under consideration, as we approach the first singular point where the velocity υ tends to zero ($X \to 0$) the ion density n tends to infinity, the value of the potential $\varphi \to -\infty$, and the electric field strength ends to $-\infty$ from the right and to $+\infty$ from the left as $X \to 0$. This means that the mathematical model must have a negative concentrated charge on the surface $X = 0$. The solid curves in Fig. 2 represent $u(X)$, $n(X)$ and $\varphi(X)$ for the given soliton particle-like solution. The solution can be interpreted both as a particle at the point $X = 0$ and as the associated wave (in the region where the value of u is significantly different from zero).

Up to the velocity of sound, where $U = c$, the soliton is similar in character (Fig. 1b). The corresponding curves for $u(X)$, $n(X)$ and $\varphi(X)$ for an acoustic particle-like soliton are represented by the dashed curves in Fig. 2.

The soliton solutions for which the ion density $n \to \infty$ as $X \to 0$ can be explained as follows. As $n \to \infty$ the initial formulation of the problem (for the model of a rarefield plasma) is no longer valid, so that a small neighbourhood of the point $X = 0$ must be removed. In addition, a solution with an infinite negative charge density on the surface $X = 0$ is physically impossible. In order to be able to discuss the solutions as a whole, a layer which is permeable to ions, through which an ion flux $n\upsilon = A$ occurs (by virtue of the first relation of system (1.12)) must be introduced into the model in a small neighbourhood of the surface $X = 0$. This layer will simultaneously act as a carrier of finite negative charge.

This very artificial formation must be used in the given one-dimensional case. In more complicated, two- or three-dimensional cases, one could perhaps construct models which are more acceptable from a physical point of view (such as small negatively charged neighbourhoods of points which are "scanned" by the flow of positive ions). However, an analysis of two- and three-dimensional solutions is beyond the scope of this paper (it would require numerical integration of initial equations). The assumption we have made that the electron density is uniform for $X \neq 0$ (instead of conforming to a Boltzmann distribution), while not affecting the form of the solutions [7, 8], does simplify the analysis considerably.

We will now describe the solution corresponding to supersonic velocity $U = 1.5$ (Fig. 1c). The supersonic stream from infinity, corresponding to the second singular point, which is a centre in this case, flows to the soliton-particle forming a leading shock wave. At the shock, the Rankine–Hugoniot relations for a direct shock are satisfied (in particular, in this case the velocity u behind the shock is defined as $u = 1/U \cong 0.667$; thus $\upsilon = u - U \cong 0.833$). The motion after that corresponds to the integral curve shown by the small crosses, up to the first singular point where there is a soliton centre which, again, we place at the origin of coordinates $X = 0$. Then from the first singular point we move along the separatrix to the third singular point. We can reach the second singular point from the third singular point in this case only by using a centred-wave solution which varies with time, and this cannot be represented by an integral curve in the phase plane. An example of the distribution $u(X)$ corresponding to a supersonic soliton-particle is represented by the dash–dot curve in Fig. 2(a) (the dashed curve here corresponds to the sonic case $U = 1$).

The solution obtained for supersonic flow is not, on the whole, steady and is essentially similar to the one-dimensional solution of a reversed isolated wave in ordinary ideal gas dynamics, when a centred rarefaction wave overtakes the shock wave and gradually reduces its intensity. In our case the rarefaction wave gradually overtakes the soliton and reduces its intensity (total mass). The leading part of the soliton does not change and can be described by the steady solution given above.

It is easy to modify the analysis of the structure of steady solutions for the case of adiabatic motion of the medium, replacing the isothermal equation of state (1.4) by the adiabatic equation in the form

$$P / P_0 = (n / n_0)^{\varkappa} \tag{2.4}$$

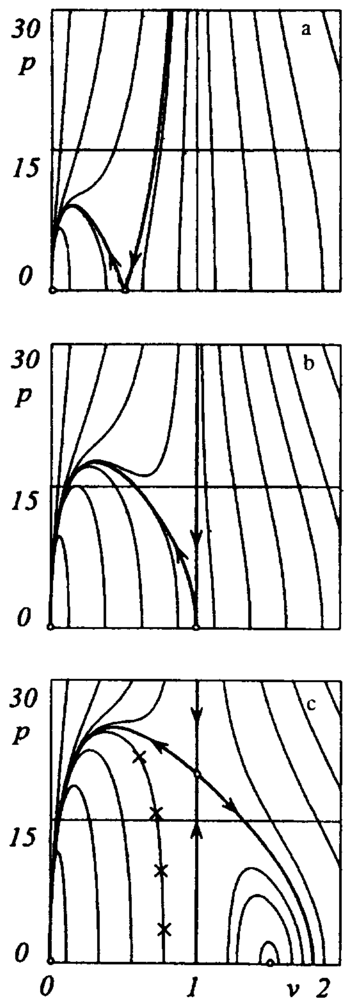

Fig. 1.

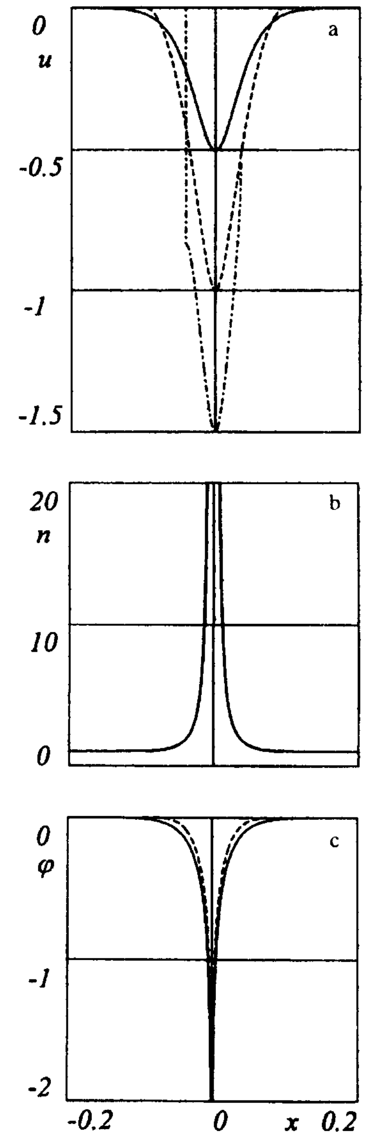

Fig. 2.

where $\varkappa$ is the adiabatic index. In this case, instead of relation (2.1), we easily obtain

$$\frac{dp}{dv}=-\frac{p^2(\varkappa^2c^2U^{\varkappa-1}+v^{\varkappa+1})+b_1^2v^{\varkappa}(v-U)}{(v^{\varkappa+1}-\varkappa c^2U^{\varkappa-1})pv} \tag{2.5}$$

The field of the integral curves given by relation (2.5) has two singular points with coordinates (2.2). Instead of the singular points with coordinates (2.3), we obtain singular points with coordinates

$$v_{3,4}=(\varkappa c^2U^{\varkappa-1})^{1/(\varkappa+1)},\quad p_{3,4}=\pm[b_1/c]\sqrt{v_{3,4}^{\varkappa}(U-v_{3,4})/(2\varkappa U^{\varkappa-1})}$$

These singular points are similar to those in the isothermal case described above.

To end this section, we note that it is possible to use the analysis in the case where the expression for **F** in the equation of motion contains a term proportional to **V**, that is, instead of (1.2) we have

$$\mathbf{F}=-\nabla\varphi_1-\nu(\mathbf{V}-\mathbf{V}_n)$$

This term allows for the interaction of charged particles with neutral particles, the mean hydrodynamic velocity of which is V_n and ν is the collision frequency of charged particles and neutral particles. In this case relation (2.1) will also contain the term $\nu p \upsilon^2$ in the numerator on the right-hand side. Some of the singular points will change in character.

3. THE STRUCTURE OF THE STEADY SOLUTIONS FOR A GAS OF NEUTRAL GRAVITATING PARTICLES

We will now consider the steady solutions of system (1.10), (1.12). The constant A, which represents the mass flow of gas, will be assumed to be non-zero. Proceeding as in the previous section, we arrive at a first-order equation similar to (2.1) but with b_1^2 replaced by $-b_2^2$ and U replaced by A. The singular points are similar to (2.2), (2.3).

The first singular point ($\upsilon_1 = 0, p_1 = 0$) is a degenerate node. The integral curves for $\upsilon \leqslant 0$ move to that point, touching the vertical axis. The second singular point ($\upsilon_2 = A, p_2 = 0$) is a centre when $A^2 < c^2$ and a saddle when $A^2 > c^2$. In the second case the angular coefficients of the characteristic directions are given by the formulae

$$k_{1,2} = \pm b_2 / \sqrt{A^2 - c^2}$$

Near this singular point, we have the following asymptotic formulae:
for $A^2 < c^2$

$$p^2 = C\exp[-C_1(\upsilon - A)] - C_2[C_1(\upsilon - A) - 1]$$

$$C = \text{const}, C_1 = (A^2 + c^2)/(2A(A^2 - c^2))$$

$$C_2 = (b_2 A^2(A^2 - c^2))/(A^2 + c^2)$$

for $A^2 > c^2$

$$p^2 = \frac{b_2^2}{a^2 - c^2}(\upsilon - A)^2$$

The third and fourth singular points ($\upsilon_{3,4} = c, p_{3,4} = \pm b_2 \sqrt{((c - A)/(2c))}$) are saddles for $A < c$ or else have imaginary coordinates. Motion towards the singular points is along the vertical separatrix $\upsilon = c$. Motion away from them is along the separatrix with angular coefficient

$$k_{3,4} = \pm[b_2 /(6c^3)]\sqrt{(2c)/(c - A)}$$

We will not consider the last singular point with negative abscissa $\upsilon = -c$

Figure 3 shows the pattern of integral curves in the phase plane (υ, p) for a low gas flow rate $A = 10^{-4}$. Motion along one of the closed integral curves around the second singular point of centre type gives a solution in the form of a period cnoidal wave. As an example, Fig. 4 shows the distribution of the density n along the coordinate x (for $U = 0$). The distributions of the velocity u and potential φ are similar. The value of the parameter A can be as small as required, allowing us to assume that when $A = 0$ there is a periodic solution with concentrated masses on surfaces passing through the peak points of the cnoidal wave.

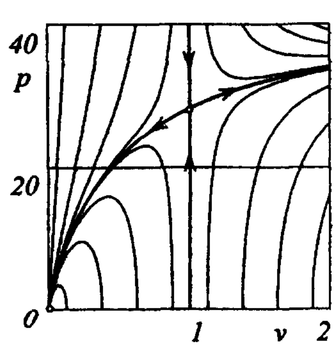

Fig. 3.

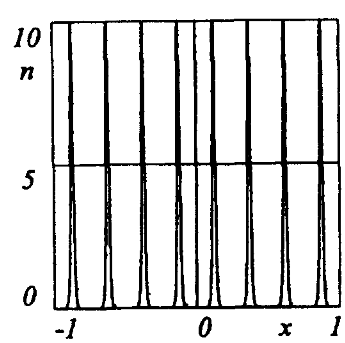

Fig. 4.

4. THE STRUCTURE OF THE STEADY SOLUTIONS IN THE CASE OF SPHERICAL SYMMETRY

Spherically symmetric steady solutions of the equations of a medium of charged particles with Coulomb interaction describe the screening of the point charge by particles of the medium with opposite-sign charges and are fairly familiar. In particular, Poisson's equations for the case of an isothermal and adiabatic plasma are given in [8]. Note also the paper [12], which gives some approximate and numerical solutions of the screening problem for the one-dimensional case.

We will make a detailed analysis of the spherically symmetric flows of a medium of neutral gravitating particles. The steady solutions (for the case $U = 0$) are described by the system of equation derived from (1.1) and (1.10)

$$u\frac{du}{dr} = -\frac{c^2}{n}\frac{dn}{dr} + \frac{d\varphi_2}{dr} \tag{4.1}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi_2}{dr}\right) = b_2^2(1-n)$$

where r is the radial coordinate.

Eliminating the density and potential from system (4.1), we obtain the equation

$$\frac{dp}{dr} = -\frac{p^2r^2(u^2+c^2)+2pru(u^2-c^2)-2c^2u^2-b_2^2u(r^2u-A)}{(u^2-c^2)ur^2} \tag{4.2}$$

where $p = du/dr$.

We know that when there is a mass source at the centre of symmetry, a steady solution can exist outside a sphere of a certain radius $r_{\min}$. We integrated Eq. (4.2) by the Runge–Kutta method, starting from a value of the coordinate $r = r_0$ where the boundary values for υ and p were assigned. Different values of r_0 and the boundary parameters correspond to different flows with the same gas flow rate but possible different total parameters.

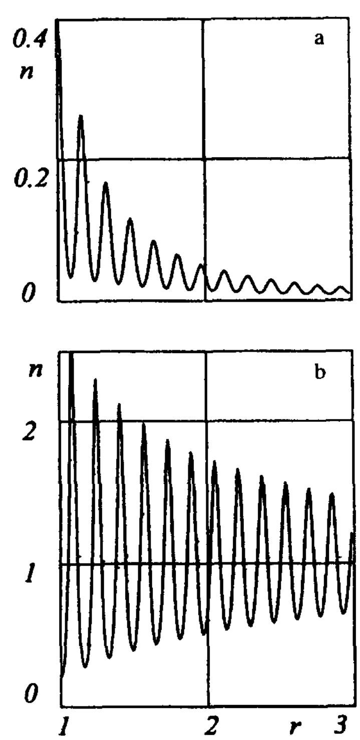

Fig. 5.

Figure 5 shows the distributions of the particle velocity and density for $A = 0$ for the boundary parameters $r_0 = 1$, $u = 0.5$ and $p = 0$.

5. CONCLUSION

The existence of steady soliton solutions with charge and mass concentrated at their centre has been demonstrated only in the case of a one-dimensional hydrodynamic model of a medium of charged particles with an intrinsic Coulomb-type force field. The solutions exist over the entire possible range of velocities of motion of the solitons (from sub- to supersonic velocities). In the supersonic range, a leading shock wave arises ahead of the soliton. In the given hydrodynamic formulation, if there is an intrinsic field of gravitating particles, periodic solutions of cnoidal wave type are obtained. The analysis demonstrates clearly which terms of the initial equations are responsible for the appearance of singular points in the phase plane and the realization of the given solutions. In subsonic cases, these solutions are symmetric (or periodic), due to the fact that the effects of non-linearity are balanced by effects associated with the existence of intrinsic force fields.

REFERENCES

1. KARPMAN, V. I., *Nonlinear Waves in Dispersive Media*. Nauka, Moscow, 1973.
2. LONGREN, K. and SCOTT, A. (Eds), *Solitons in Action*. Academic Press, New York, 1978.
3. ZEITUNYAN, R. Kh., Nonlinear long waves on the water surface and solitons. *Uspekhi Fiz. Nauk*, **165**, 1995, 12, 1403–1456.
4. BEREZIN, Yu. A. and FEDORUK, M. P., *The Simulation of Unsteady Plasma Processes*. Nauka, Novosibirsk.
5. ZAKHAROV, V. Ye., MANAKOV, S. V., NOVIKOV S. P. and PITAYEVSKII, L. P., *Theory of Solitons*. Nauka, Moscow, 1980.
6. BULLOUGH, R. K. and CAUDREY, P. J., *Solitons*. Springer, Berlin, 1980.
7. IVANOV, M. Ya., On a class of soliton solutions of the hydrodynamic equations of ion motion in a homogeneous plasma with zero applied fields. *Fiz. Plamy*, 1982, **8**, 3, 607–612.
8. IVANOV, M. Ya., On the theoretical possibility of restraining a dense adiabatic plasma by a self-consistent electrostatic field in the form of subsonic solitons. *Zh. Tekh. Fiz.*, 1983, **53**, 2, 387–390.
9. ALEKSANDROV, A. F., BOGDANKEVICH, L. S. and RUKHADZE, A. A., *The Fundamentals of Plasma Electrodynamics*. Vyssh. Shkola, Moscow, 1988.
10. IVANOV, M. Ya., The analysis of ion solitons in a plasma without a magnetic field. *Fiz. plazmy*, 1982, **8**, 2, 384–389.
11. A repulsive force in the Universe. *Physics News Update. The American Institute of Physics Bulletin of Physics News*, 1998, 361.
12. CLEMENTE, R. A. and MARTIN, P., Non-linear unidimensional Debye screening in plasmas. *J. Phys. Soc. Japan*, 1992, **61**, 6, 1969–1972.

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